On $r$-Dynamic Coloring of Some Special Graphs Operations

I.H. Agustin$^{1,2}$, Dafik$^{1,3}$, N.I. Wulandari$^{1,2}$

$^1$CGANT - University of Jember
$^2$Department of Mathematics Education - University of Jember
$^3$Department of Information System - University of Jember

Hestyarin@gmail.com; d.dafik@gmail.com; nuricswulandari.2018@gmail.com

Abstract

Let $G$ be a simple, connected and undirected graph. Let $r, k$ be natural number. By a proper $k$-coloring of a graph $G$, we mean a map $c: V(G) \rightarrow S$, where $|S| = k$, such that any two adjacent vertices receive different colors. An $r$-dynamic $k$-coloring is a proper $k$-coloring $c$ of $G$ such that $|c(N(v))| \geq \min\{r, d(v)\}$ for each vertex $v$ in $V(G)$, where $N(v)$ is the neighborhood of $v$ and $c(S) = \{c(v) : v \in S\}$ for a vertex subset $S$. The $r$-dynamic chromatic number, written as $\chi_r(G)$, is the minimum $k$ such that $G$ has an $r$-dynamic $k$-coloring. The 1-dynamic chromatic number of graph is equal to its chromatic number, denoted by $\chi(G)$, and the 2-dynamic chromatic number of graph has been studied under the name a dynamic chromatic number, denoted by $\chi_d(G)$. By simple observation it is easy to see that $\chi_r(G) \leq \chi_{r+1}(G)$, for example $\chi(C_6) = 2$, but $\chi_d(C_6) = 3$. In this paper we will show the exact values of some graph operation of special graphs.

Keywords: $r$-dynamic coloring, $r$-dynamic chromatic number, graph operations.

Mathematic Subject Classification: 05C15

Introduction

Let $G = (V, E)$ be a simple, connected and undirected graph with vertex set $V$ and edge set $E$, and $d(v)$ be a degree of any $v \in V(G)$. The maximum degree and the minimum degree of $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. By a proper $k$-coloring of a graph $G$, we mean a map $c: V(G) \rightarrow S$, where $|S| = k$, such that any two adjacent vertices receive different colors. An $r$-dynamic $k$-coloring is a proper $k$-coloring $c$ of $G$ such that $|c(N(v))| \geq \min\{r, d(v)\}$ for each vertex $v$ in $V(G)$, where $N(v)$ is the neighborhood of $v$ and $c(S) = \{c(v) : v \in S\}$ for a vertex subset $S$. The $r$-dynamic chromatic number, written as $\chi_r(G)$, is the minimum $k$ such that $G$ has an $r$-dynamic $k$-coloring. Note that the 1-dynamic chromatic number of graph is equal to its chromatic number, denoted by $\chi(G)$, and the 2-dynamic chromatic number was introduced by Montgomery [5] under the name a dynamic chromatic number, denoted by $\chi_d(G)$. He conjectured $\chi_d(G) \leq \chi(G) + 2$ when $G$ is regular, which remains open. Akbari et. al [4] proved Montgomery’s conjecture for bipartite regular graphs. Lai, Montgomery, and Poon [6] proved $\chi_2(G) \leq \Delta(G) + 1$ when $\Delta(G) \geq 3$ and no component is the 5-cycle $C_5$. Kim et. al proved $\chi_d(G) \leq 4$ for planar $G$ with girth at least 7, and $\chi_d(G) \leq k$ when $k \geq 4$ and $G$ has maximum average degree at most $\frac{2k}{k+2}$ (both results are sharp). Kim et. al [3] proved $\chi_2(G) \leq 4$ when $G$ is planar and no component is $C_5$ and also $\chi_d \leq 5$ whenever $G$ is planar.
Obviously, $\chi(G) \leq \chi_2(G)$, but it was shown in [6] that the difference between the chromatic number and the dynamic chromatic number can be arbitrarily large. However, it was conjectured that for regular graphs the difference is at most 2. Also, it was proved in [6] that, if $G$ is a bipartite $k$-regular graph, $k \geq 3$ and $n < 2^k$, then $\chi_2 \leq 4$. Some properties of dynamic coloring were studied in [3, 4, 6]. It was proved in [8] that, for a connected graph $G$, if $\Delta \leq 3$, then $\chi_2(G) \leq 4$ unless $G = C_5$, in which case $\chi_2(C_5) = 5$ and if $\Delta \geq 4$ then $\chi(G) \leq \Delta + 1$.

A Useful Lemma

The following lemmas are useful for determining the dynamic coloring of graphs. This lemma characterize the upper bound in term of the diameter of graph.

\begin{itemize}
  \item [\textbf{Teorema 1}] [7] If $\text{diam}(G) = 2$, then $\chi_2(G) \leq \chi(G) + 2$, with equality only when $G$ is a complete bipartite graph or $C_5$.
  \item [\textbf{Teorema 2}] [7] If $G$ is a $k$-chromatic graph with diameter at most 3, then $\chi_2(G) \leq 3k$, and this bound is sharp when $k \geq 2$.
\end{itemize}

In term of the maximum degree of graph, the $r$-dynamic of graph satisfies as follows

\begin{itemize}
  \item [\textbf{Observation 1}] [7] $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$, and this is sharp. If $\Delta(G) \leq r$ then $\chi_r(G) = \min\{\Delta(G), r\}$.
  \item [\textbf{Teorema 3}] [7] $\chi_r(G) \leq r\Delta(G) + 1$, with equality for $r \geq 2$ if and only if $G$ is $r$-regular with diameter 2 and girth 5.
\end{itemize}

Let $G^2$ denote the graph obtained from $G$ by adding edges joining nonadjacent vertices that have a common neighbor, Jahanbekam et. al [7] proved the following.

\begin{itemize}
  \item [\textbf{Observation 2}] [7] $\chi(G) \leq \chi_d(G) \leq \chi_3(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi(G^2)$.
\end{itemize}

The last for graph operations of cartesian product, we have the following

\begin{itemize}
  \item [\textbf{Teorema 4}] [7] If $\delta(G) \geq r$ then $\chi_r(G \square H) = \max\{\chi(G), \chi(H)\}$.
\end{itemize}
The Results

Now, we are ready to show our results on $r$-dynamic coloring for some special graph operations. Apart from showing the $r$-dynamic chromatic number we also show the colors $c(v \in V(G))$ for clarity. Some graph operations found in this paper are $W_n + P_m, W_n \square P_m, W_n \otimes P_m, W_n[P_m], W_n \otimes P_m, shcak(S_n + P_m, v, s), amal(W_n + P_m, v, s)$

$\diamond \textbf{Theorem 5}$ Let $G$ be a joint $W_n$ and $P_m$. For $n \geq 3$ dan $m \geq 2$, the $r$-dynamic chromatic number of $G$ is

$$\chi(G) = \chi_d(G) = \chi_3(G) = \chi_4(G)$$

\begin{align*}
\begin{cases}
5, & \text{for } n \text{ even} \\
6, & \text{for } n \text{ odd}
\end{cases}
\end{align*}

\textbf{Proof.} The graph $W_n + P_m$ is a connected graph with vertex set $V(W_n + P_m) = \{A_i, x_i, y_j; 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(P_n + C_m) = \{A_i x_i; 1 \leq i \leq n\} \cup \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{A y_j; 1 \leq j \leq m\} \cup \{y_j y_{j+1}; 1 \leq j \leq m-1\}$. Thus $p = |V(W_n + P_m)| = n + m + 1, q = |E(G)| = nm + 2n + 2m - 1$ and $\Delta(W_n + P_m) = m + n$. By Observation 1, the lower bound of $r$-dynamic chromatic number $\chi_r(W_n + P_m) \geq \min\{\Delta(W_n + P_m), r\} + 1 = \{m + n, r\} + 1$. Define the vertex coloring $c : V(W_n + P_m) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows:

For $n$ even
\begin{align*}
c(x_i) = \begin{cases}
1, & 1 \leq i \leq n, \ i \text{ even} \\
2, & 1 \leq i \leq n, \ i \text{ odd}
\end{cases}
\end{align*}
\begin{align*}
c(y_j) = \begin{cases}
4, & 1 \leq j \leq m, \ j \text{ odd} \\
5, & 1 \leq j \leq m, \ j \text{ even}
\end{cases}
\end{align*}

For $n$ odd
\begin{align*}
c(x_i) = \begin{cases}
1, & 1 \leq i \leq n - 1, \ i \text{ odd} \\
2, & 1 \leq i \leq n - 1, \ i \text{ even} \\
4, & i = n
\end{cases}
\end{align*}
\begin{align*}
c(y_j) = \begin{cases}
5, & 1 \leq j \leq m, \ j \text{ odd} \\
6, & 1 \leq j \leq m, \ j \text{ even}
\end{cases}
\end{align*}

It is easy to see that $c : V(W_n + P_m) \rightarrow \{1, 2, \ldots, 4\}$ and $c : V(W_n + P_m) \rightarrow \{1, 2, \ldots, 5\}$, for $n$ even and odd respectively, is proper coloring. Thus, $\chi(W_n + P_m) = 5$ and $\chi(W_n + P_m) = 6$, for $m$ even and odd respectively. By definition, since $\min\{|c(N(v))|$, for every $v \in V(W_n + P_m)\} = 4$, it implies $\chi(W_n + P_m) = \chi_d(W_n + P_m) = \chi_3(W_n + P_m) = \chi_4(W_n + P_m)$. It completes the proof. \hfill $\square$

\textbf{Open Problem 1} Let $G$ be a joint $W_n$ and $P_m$. For $n \geq 2$ and $m \geq 3$, determine the $r$-dynamic chromatic number of $G$ when $r \geq 5$. 
\textbf{Theorem 6} Let $G$ be a cartesian product of $W_n$ and $P_m$. For $n \geq 3$ and $m \geq 2$, the $r$-dynamic chromatic number of $G$ is

$$\chi(G) = \begin{cases} 3, & \text{for } n \text{ even} \\ 4, & \text{for } n \text{ odd} \end{cases}$$

\textbf{Proof.} The graph $W_n \square P_m$ is a connected graph with vertex set $V(W_n \square P_m) = \{A_j, x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(W_n \square P_m) = \{A_j A_{j+1}; 1 \leq i \leq m - 1\} \cup \{A_j x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\} \cup \{x_{i,j} x_{i+1,j}; 1 \leq i \leq n - 1; 1 \leq j \leq m\} \cup \{x_{1,j} x_{n,j}; 1 \leq j \leq m\} \cup \{x_{i,j} x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m - 1\}$. Thus $|V(W_n \square P_m)| = nm + m$ and $|E(W_n \square P_m)| = 4nm$ and $\Delta(W_n \square P_m) = 6$. By Observation 1, the lower bound of $r$-dynamic chromatic number $\chi_r(W_n \square P_m) \geq \min\{\Delta(W_n \square P_m), r\} + 1 = \{6, r\} + 1$. Define the vertex colouring $c : V(W_n \square P_m) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows:

$$c(A_i) = \begin{cases} 1, & 1 \leq i \leq n, \ i \text{ odd} \\ 2, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

For $n$ even

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n, \ i \text{ even}, \ 1 \leq j \leq m, \ j \text{ even} \\ 2, & 1 \leq i \leq n, \ i \text{ odd}, \ 1 \leq j \leq m, \ j \text{ odd} \\ 3, & 1 \leq i \leq n, \ i \text{ even}, \ 1 \leq j \leq m, \ j \text{ odd}, \ 1 \leq i \leq n, \ i \text{ odd} \\ 1 \leq j \leq m, \ j \text{ odd} \end{cases}$$

For $n$ odd

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n, \ i \text{ even}, \ 1 \leq j \leq m, \ j \text{ odd} \\ 2, & 1 \leq i \leq n - 1, \ i \text{ odd}, \ 1 \leq j \leq m, \ j \text{ even} \\ 3, & 1 \leq i \leq n - 1, \ i \text{ odd}, \ 1 \leq j \leq m, \ j \text{ odd}; i = n, 1 \leq j \leq m, \ j \text{ even} \\ 4, & 1 \leq i \leq n, \ i \text{ even}, \ 1 \leq j \leq m, \ j \text{ even}; i = n, 1 \leq j \leq m, \ j \text{ odd} \end{cases}$$

It is easy to see that $c : V(W_n \square P_m) \rightarrow \{1, 2\}$ and $c : V(W_n \square P_m) \rightarrow \{1, 2, 3, 4\}$, for $n$ even and odd respectively, is proper coloring. By definition, since $\min\{|c(N(v))|, \text{ for every } v \in V(W_n \square P_m)\} = 1$, thus we only have $\chi(W_n \square P_m) = 3$ and $\chi(W_n \square P_m) = 4$, for $n$ even and odd respectively.\hfill \Box

\textbf{Open Problem 2} Let $G$ be a cartesian product of $W_n$ and $P_m$. For $n \geq 3$ and $m \geq 2$, determine the $r$-dynamic chromatic number of $G$ when $r \geq 3$. 
\textbf{Teorema 7} Let $G$ be a tensor product of $W_n$ and $P_m$. For $n \geq 3$ dan $m \geq 2$, the $r$-dynamic chromatic number of $G$ is $\chi(W_n \otimes P_m) = 2$

\textbf{Proof.} The graph $W_n \otimes P_m$ is a connected graph with vertex set $V = \{A_j, x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ dan $E = \{A_jx_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m - 1\}$ $\cup \{A_jx_{i,j-1}; 1 \leq i \leq n; 1 \leq j \leq m - 1\}$ $\cup \{x_{i,j}x_{i+1,j+1}; 1 \leq i \leq n - 1; 1 \leq j \leq m - 1\} \cup \{x_{i,j}x_{i-1,j+1}; 2 \leq i \leq n - 1; 1 \leq j \leq m - 1\}$ $\cup \{x_{n,j}x_{i,j+1}; 1 \leq j \leq m - 1\}$ $\cup \{x_{n,j}x_{i,j-1}; 1 \leq j \leq m - 1\}$ $\cup \{x_{i,j}x_{i,j+1}; 1 \leq j \leq m - 1\}$ $\cup \{x_{i,j}x_{i+1,j}; 1 \leq j \leq m - 1\} = 2m - 4n$ dan $\Delta(W_n \otimes P_m) = 2n$. By Observation 1, the lower bound of $r$-dynamic chromatic number $\chi_r(W_n \otimes P_m) \geq \min\{\Delta(W_n \otimes P_m), r\} + 1 = \{2n, r\} + 1$. Define the vertex coloring $c : V(W_n \otimes P_m) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows:

$$c(A_j) = \begin{cases} 
1, & 1 \leq j \leq m, \ j \text{ odd} \\
2, & 1 \leq j \leq m, \ j \text{ even}
\end{cases}$$

It is easy to see that $c : V(W_n \otimes P_m) \rightarrow \{1, 2\}$ is proper coloring. By definition, since $\min\{|c(N(v))|, \text{ for every } v \in V(W_n \otimes P_m)\} = 1$, thus we only have $\chi(W_n \otimes P_m) = 2$. \hfill \square

\textbf{Open Problem 3} Let $G$ be a tensor product of $W_n$ and $P_m$. For $n \geq 3$ and $m \geq 2$, determine the $r$-dynamic chromatic number of $G$ when $r \geq 2$.

\textbf{Teorema 8} Let $G$ be a composition of graph $W_n$ on $P_m$. For $n \geq 3$ dan $m \geq 2$, the $r$-dynamic chromatic number of $G$ is

$$\chi(W_n[P_m]) = \begin{cases} 
6, & \text{for } n \text{ even} \\
8, & \text{for } n \text{ odd}
\end{cases}$$

\textbf{Proof.} The graph $W_n[P_m]$ is a connected graph with vertex set $V(W_n[P_m]) = \{A_j, x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(W_n[P_m]) = \{A_jx_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m - 1\} \cup \{A_jx_{i,j-1}; 1 \leq i \leq n; 1 \leq j \leq m - 1\}$ $\cup \{x_{i,j}x_{i+1,j+1}; 1 \leq i \leq n - 1; 1 \leq j \leq m - 1\} \cup \{x_{i,j}x_{i-1,j+1}; 2 \leq i \leq n - 1; 1 \leq j \leq m - 1\}$ $\cup \{x_{n,j}x_{i,j+1}; 1 \leq j \leq m - 1\} \cup \{x_{n,j}x_{i,j-1}; 1 \leq j \leq m - 1\}$ $\cup \{A_jx_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ $\cup \{x_{i,j}x_{i+1,j}; 1 \leq i \leq n - 1; 1 \leq j \leq m\}$ $\cup \{x_{i,j}x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m - 1\} = 2m - 4n$. By Observation 1, the lower bound of $r$-dynamic chromatic number $\chi_r(W_n[P_m]) \geq \min\{\Delta(W_n[P_m]), r\} + 1 = \{2n + 2, r\} + 1$. Define the vertex coloring $c : V(W_n[P_m]) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows:
For $n$ even and $n$ odd
\[
c(A_j) = \begin{cases} 
1, & 1 \leq j \leq m, \; j \text{ odd} \\
4, & 1 \leq j \leq m, \; j \text{ even}
\end{cases}
\]

For $n$ even
\[
c(x_{i,j}) = \begin{cases} 
2, & 1 \leq i \leq n - 1, \; i \text{ odd}; \; 1 \leq j \leq m, \; j \text{ odd} \\
3, & 1 \leq i \leq n, \; i \text{ even}; \; 1 \leq j \leq m, \; j \text{ odd} \\
5, & 1 \leq i \leq n, \; i \text{ even}; \; 1 \leq j \leq m, \; j \text{ even} \\
6, & 1 \leq i \leq n, \; i \text{ even}; \; 1 \leq j \leq m, \; j \text{ even}
\end{cases}
\]

For $n$ odd
\[
c(x_{i,j}) = \begin{cases} 
2, & 1 \leq i \leq n - 1, \; i \text{ odd}; \; 1 \leq j \leq m, \; j \text{ odd} \\
3, & 1 \leq i \leq n, \; i \text{ even}; \; 1 \leq j \leq m, \; j \text{ odd} \\
5, & 1 \leq i \leq n - 1, \; i \text{ odd}; \; 1 \leq j \leq m, \; j \text{ even} \\
6, & 1 \leq i \leq n, \; i \text{ even}; \; 1 \leq j \leq m, \; j \text{ even} \\
7, & i = n, \; 1 \leq j \leq m, \; j \text{ odd} \\
8, & i = n, \; 1 \leq j \leq m, \; j \text{ even}
\end{cases}
\]

It is easy to see that $c: V(W_n[P_m]) \rightarrow \{1, 2, \ldots, 6\}$ and $c: V(W_n[P_m]) \rightarrow \{1, 2, \ldots, 8\}$, for $n$ even and odd respectively, is proper coloring. Thus, $\chi(W_n[P_m]) = 6$ and $\chi(W_n[P_m]) = 8$, for $n$ even and odd respectively. By definition, since $\min\{|c(N(v))|\}$, for every $v \in V(W_n[P_m]) = 5$, it implies $\chi(W_n[P_m]) = \chi_d(W_n[P_m]) = \chi_3(W_n[P_m]) = \chi_4(W_n[P_m]) = \chi_5(W_n[P_m])$. It completes the proof. \hfill \Box

**Open Problem 4** Let $G$ be a composition of $W_n$ and $P_m$. For $n \geq 3$ and $m \geq 2$, determine the $r$-dynamic chromatic number of $G$ when $r \geq 6$.

\diamond **Theorem 9** Let $G$ be a crown product of $W_n$ on $P_m$. For $n \geq 3$ dan $m \geq 2$, the $r$-dynamic chromatic number of $G$ is

\[
\chi(W_n \odot P_m) = \chi_d(W_n \odot P_m) = \begin{cases} 
3, & \text{for } n \text{ even} \\
4, & \text{for } n \text{ odd}
\end{cases}
\]

**Proof.** The graph $W_n \odot P_m$ is a connected graph with vertex set $V(W_n \odot P_m) = \{A, x_i, x_{i,j}, y_j; 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(W_n \odot P_m) = \{Ax_i; 1 \leq i \leq n\} \cup \{x_{i,x_{i+1}}; 1 \leq i \leq n - 1\} \cup \{Ay_j; 1 \leq j \leq m\} \cup \{y_jy_{j+1}; 1 \leq j \leq m - 1\} \cup \{x_1x_n\} \cup \{x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m\} \cup \{x_{i,x_{i,j+1}}; 1 \leq i \leq n; 1 \leq j \leq m - 1\}$. Thus $|V(W_n[P_m])| = nm + n + m + 1$ and $|E(W_n \odot P_m)| = 2mn + n + 2m - 1$ and $\Delta(W_n \odot P_m) = n + m$. By Observation 1, the lower bound of $r$-dynamic
chromatic number $\chi_r(W_n \odot P_m) \geq min\{\Delta(W_n \odot P_m), r\} + 1 = \{n + m, r\} + 1$.

Define the vertex coloring $c : V(W_n \odot P_m) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows: $A = 4$ and

$$c(y_j) = \begin{cases} 1, & 1 \leq j \leq m, \text{j even} \\ 3, & 1 \leq j \leq m, \text{j odd} \end{cases}$$

For $n$ even

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n, \text{i odd}; 1 \leq j \leq m, \text{j even} \\ 2, & 1 \leq i \leq n, \text{i even}; 1 \leq j \leq m, \text{j even} \\ 3, & 1 \leq j \leq m, \text{j odd}; 1 \leq i \leq n \end{cases}$$

$$c(x_i) = \begin{cases} 1, & 1 \leq i \leq n, \text{i even} \\ 2, & 1 \leq i \leq n, \text{i odd} \end{cases}$$

For $n$ odd

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n, \text{i odd}; 1 \leq j \leq m, \text{j even} \\ 2, & 1 \leq i \leq n, \text{i even}; 1 \leq j \leq m, \text{i even} \\ 3, & 1 \leq j \leq m-1, \text{j even}; 1 \leq i \leq n-1 \\ 4, & 1 \leq j \leq m, \text{j odd}; i = n \end{cases}$$

$$c(x_i) = \begin{cases} 1, & 1 \leq i \leq n-1, \text{i even} \\ 2, & 1 \leq i \leq n-1, \text{i odd} \\ 3, & i = n \end{cases}$$

It is easy to see that $c : V(W_n \odot P_m) \rightarrow \{1, 2, \ldots, 3\}$ and $c : V(W_n \odot P_m) \rightarrow \{1, 2, \ldots, 4\}$, for $n$ even and odd respectively, is proper coloring. Thus, $\chi(W_n \odot P_m) = 3$ and $\chi(W_n \odot P_m) = 4$, for $n$ even and odd respectively. By definition, since $min\{|c(N(v))|$, for every $v \in V(W_n \odot P_m)\} = 2$, it implies $\chi(W_n \odot P_m) = \chi_d(W_n \odot P_m)$. It completes the proof. $\square$

**Open Problem 5** Let $G$ be a crown product of $W_n$ on $P_m$. For $n \geq 3$ dan $m \geq 2$, determine the $r$-dynamic chromatic number of $G$ when $r \geq 3$.

**Teorema 10** Let $G$ be a shackle of joint $S_n$ and $P_m$. For $n \geq 3$ dan $m \geq 2$, the $r$-dynamic chromatic number of $G$ is

$$\chi(\text{shack}(S_n + P_m, v, s)) = \chi_d(\text{shack}(S_n + P_m, v, s)) = 4$$

**Proof.** The shackle of joint $S_n$ and $P_m$, denoted by $\text{shack}(S_n + P_m, v, s)$, is a connected graph with vertex set $V = \{A_k, x_i^k, x_i^j, y_j^k, p; 1 \leq i \leq n; 1 \leq j \leq$
m; 1 \leq k \leq s\}) \text{ and } E = \{A_k x^{k}_i; 1 \leq i \leq n - 1; 1 \leq k \leq s\} \cup \{A_k x^{k+1}_i; 1 \leq k \leq s\} \cup \{A_k y^{k}_j; 1 \leq j \leq m - 1; 1 \leq k \leq s\} \cup \{A_k y^{k+1}_j; 1 \leq j \leq m; 1 \leq k \leq s\} \cup \{x^{k}_i y^{k+1}_j; 1 \leq i \leq n - 1; 1 \leq j \leq m; 1 \leq k \leq s\} \cup \{x^{k+1}_i y^{k+1}_j; 1 \leq j \leq m; 1 \leq k \leq s - 1\} \cup \{y^{k+1}_j; 1 \leq j \leq m\}. \text{ Thus } |V(\text{shack}(S_n + P_m, v, s))| = nr + mr + 1 \text{ and } |E(\text{shack}(S_n + P_m, v, s))| = 2nms + ns + 2ms - s \text{ and } \Delta(\text{shack}(S_n + P_m, v, s)) = 6. \text{ By Observation 1, the lower bound of } r\text{-dynamic chromatic number } \chi_r(\text{shack}(S_n + P_m, v, s)) \geq \min\{\Delta(\text{shack}(S_n + P_m, v, s)), r\} + 1 = \{6, r\} + 1.

Define the vertex coloring \( c : V(\text{shack}(S_n + P_m, v, s)) \rightarrow \{1, 2, \ldots, k\} \) for \( n \geq 3 \) and \( m \geq 2 \) as follows: \( c(A^k) = 4 \)

\[
c(x^k_i) = \begin{cases} 3, & 1 \leq i \leq n - 1; 1 \leq k \leq s \\
1, & 1 \leq j \leq m, j \text{ odd}; 1 \leq k \leq s \\
2, & 1 \leq j \leq m, j \text{ even}; 1 \leq k \leq s
\end{cases}
\]

It is easy to see that \( c : V(\text{shack}(S_n + P_m, v, s)) \rightarrow \{1, 2, \ldots, 4\} \) is proper coloring. Thus, \( \chi(\text{shack}(S_n + P_m, v, s)) = 4 \). By definition, since \( \min\{|c(N(v))| \} \), for every \( v \in V(\text{shack}(S_n + P_m, v, s)) \) = \( 3 \), it implies \( \chi(\text{shack}(S_n + P_m)) = \chi(d(\text{shack}(S_n + P_m))) = \chi_3(\text{shack}(S_n + P_m)) \). It completes the proof. \( \square \)

**Open Problem 6** Let \( G \) be a shacke of joint \( S_n \) and \( P_m \). For \( n \geq 3 \) and \( m \geq 2 \), determine the \( r\)-dynamic chromatic number of \( G \) when \( r \geq 4 \).

\( \diamond \) **Theorem 11** Let \( G \) be an amalgamation of joint \( W_n \) and \( P_m \). For \( n \geq 2 \) and \( m \geq 3 \), the \( r\)-dynamic chromatic number of \( G \) is

\[
\chi(\text{Amal}(W_n + P_m, v, s)) = \begin{cases} 5, & \text{for } n \text{ even} \\
6, & \text{for } n \text{ odd}
\end{cases}
\]

**Proof.** Amalgamation of joint \( W_n \) and \( P_m \), denoted by \( \text{amal}(W_n + P_m, v, s) \), is a connected graph with vertex set \( V(\text{amal}(W_n + P_m, v, s)) = \{A_k, x^{k}_i, y^{k}_j, y_1; 1 \leq i \leq n; 2 \leq j \leq m; 1 \leq k \leq r\} \) and \( E(\text{amal}(W_n + P_m, v, s)) = \{A_k x^{k}_i; 1 \leq i \leq n; 1 \leq k \leq s\} \cup \{A_k y^{k}_j; 2 \leq j \leq m - 1; 1 \leq k \leq s\} \cup \{x^{k}_i x^{k+1}_i; 1 \leq i \leq n - 1; 1 \leq k \leq s\} \cup \{x^{k+1}_i x^{k+1}_i; 1 \leq i \leq n; 2 \leq j \leq m - 1; 1 \leq k \leq s\} \cup \{x^{k+1}_i y^{k+1}_j + 1^k; 1 \leq i \leq n; 2 \leq j \leq m - 1; 1 \leq k \leq s\} \cup \{y^{k+1}_j y^{k+1}_j; 1 \leq k \leq s\} \cup \{y^{k+1}_j y^{k+1}_j + 1^k; 2 \leq j \leq m - 1; 1 \leq k \leq s\} \cup \{y_1 y^{k+1}_j; 1 \leq k \leq s\}. \text{ Thus } |V(\text{amal}(W_n + P_m, v, s))| = ns + ms + 1 \text{ and } |E(\text{amal}(W_n + P_m, v, s))| = 2nm + n + 2m - 2 \text{ and } \Delta(\text{amal}(W_n + P_m, v, s)) = (n + m)r. \text{ By Observation 1, the lower bound of } r\text{-dynamic chromatic number } \chi_r(\text{amal}(W_n + P_m, v, s)) \geq \min\{\Delta(\text{amal}(W_n + P_m, v, s)), r\} + 1 = \{3(n + m)s, r\} + 1. \text{ Define the vertex}
coloring $c : V(amal(W_n + P_m, v, s)) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows: $c(A^k) = 1$ and $c(y_1) = 4$

$$c(y_j^k) = \begin{cases} 
4, & 2 \leq j \leq m, \ j \text{ odd; } 1 \leq k \leq s \\
5, & 2 \leq j \leq m, \ j \text{ even; } 1 \leq k \leq s 
\end{cases}$$

For $n$ even

$$c(x_i^k) = \begin{cases} 
2, & 1 \leq i \leq n - 1, \ j \text{ odd; } 1 \leq k \leq s \\
3, & 1 \leq i \leq n - 1, \ j \text{ even; } 1 \leq k \leq s 
\end{cases}$$

$$c(p) = 3$$

For $n$ odd

$$c(x_i^k) = \begin{cases} 
2, & 1 \leq i \leq n - 1, \ j \text{ odd; } 1 \leq k \leq s \\
3, & 1 \leq i \leq n - 1, \ i \text{ even; } 1 \leq k \leq s 
\end{cases}$$

Clearly $c : V(amal(W_n + P_m, v, s)) \rightarrow \{1, \ldots, 5\}$ and $c : V(amal(W_n + P_m, v, s)) \rightarrow \{1, \ldots, 6\}$, for $n$ even and odd respectively, are proper coloring. Thus, for $n$ even, $\chi(amal(W_n + P_m, v, s)) = 5$ and for $n$ odd, $\chi(amal(W_n + P_m, v, s)) = 6$. By definition, since $\min\{|c(N(v))|, \text{ for every } v \in V(amal(W_n + P_m, v, s))\} = 4$, it implies $\chi(amal(W_n + P_m, v, s)) = \chi_d(amal(W_n + P_m, v, s)) = \chi_3(amal(W_n + P_m, v, s)) = \chi_4(amal(W_n + P_m, v, s))$. It completes the proof. □

**Open Problem 7** Let $G$ be a shackle of joint $W_n$ and $P_m$. For $n \geq 3$ and $m \geq 2$, determine the $r$-dynamic chromatic number of $G$ when $r \geq 5$.

**Conclusions**

We have studied the $r$-dynamic coloring of some graph operations. The results show for each graph operation we have not obtained completely all values of $r$, therefore we left as an open problem for the reader.

**References**


